Hilbert's Axioms

March 26, 2013

1 Flaws in Euclid

The description of "a point between two points, line separating the plane into two sides, a segment is congruent to another segment, and an angle is congruent to another angle," are only demonstrated in Euclid's *Elements*.

2 Axioms of Betweenness

Points on line are not unrelated. We assume that there is a ternary relation among points, named as "**point B is between point A and point C**," abbreviated as

$$A * B * C$$

Given distinct collinear points A, B, C, D. We use

$$A * B * C * D$$

to denote the following simultaneous relations of betweenness

$$A * B * C, \quad A * B * D, \quad A * C * D, \quad B * C * D. \tag{1}$$

Betweenness Axiom 1 (BA1) (Collinearity and symmetrization). If $A * B * C$, then A, B, C are three distinct points all lying on the same line, and $C * B * A$.

Betweenness Axiom 2 (BA2) (Extension). Given two distinct points B and D on a line l. There exist points A, C, E lying on line l such that $A * B * D$, $B * C * D$, and $B * D * E$; see Figure 1.

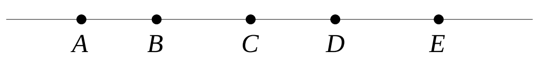

Figure 1: Betweenness Axiom 2

Betweenness Axiom 3 (BA3) (Uniqueness). Let A, B, C br three distinct points on a line. Then one and only one of the three points is between the other two.

Definition 1 (Line, segment, and ray). The line determined by two distinct points A and B is denote by

$$\overline{AB}.$$

We also use $\overline{AB}$ to denote the set of all points incident with the line determined by points A and B. A **segment** with endpoints A and B, denoted

$$AB,$$

is the set of points A, B, and all points between A and B. A **ray** emanating from a point A to another point B, denoted

$$r(A, B),$$

is the set of all points on AB and all points C such that $A * B * C$. An **open ray** emanating from a point A to another point B is the set

$$\mathring{r}(A, B) := r(A, B) \smallsetminus \{A\}.$$

Proposition 2.1. *For any two distinct points A and B,*

$$AB = r(A, B) \cap r(B, A), \quad \overline{AB} = r(A, B) \cup r(B, A).$$

Proof. Note that $AB \subseteq r(A, B) \cap r(B, A)$ by definition of segment and ray. For each point $P \in r(A, B) \cap r(B, A)$, we have $P \in r(A, B)$ and $P \in r(B, A)$. Suppose $P \notin AB$. By definition of ray, we have $A * B * P$ by $P \in r(A, B)$ and $P * A * B$ by $P \in r(B, A)$. Then A, B, P are three distinct collinear points by BA1. This is contradictory to BA3 that there is only one point of the three A, B, P between the other two.

It is clear that $r(A, B) \cup r(B, A) \subseteq \overline{AB}$. For each $P \in \overline{AB}$, if $P \in AB$, it is clear that $P \in r(A, B) \cup r(B, A)$. Assume $P \notin AB$, then A, B, P are three distinct points by BA1, and one of them is between the other two by BA3. Since P is not between A and B, we have either A is between B and P or B is between A and P. In the formal case, we have $P \in r(B, A)$; in the latter case, we have $P \in r(A, B)$. Hence $P \in r(A, B) \cup r(B, A)$. $\square$

Definition 2 (Same side and opposite side). Two points A, B not on a line l are said to be on the **same side** of l if $A = B$ or the segment AB does not meet l. Two points A, B not on a line l are said to be on **opposite sides** of l if AB does not meet l.

Betweenness Axiom 4 (BA4) (Plane separation). Let A, B, C be three distinct points not on a line l.

(i) If A, B are on the same side of l and B, C are on the same side of l, then A, C are on the same side of l.

(ii) If A, B are on opposite sides of l and B, C are on opposite sides of l, then A, C are on the same side of l.

The relation of being on the same side of a fixed line l is an equivalence relation on the set of points not on the line l, since it is reflexive, symmetric, and transitive by definition and Betweenness Axiom 4(i). Each equivalence class is called an **open half-plane** bounded by l. For each point P not on l, we denoted by

$$\mathring{H}(l, P)$$

the open half-plane that contains P. The set

$$H(l, P) := \mathring{H}(l, P) \cup l$$

is called a **half-plane** (or **closed half-plane**) bounded by l.

Corollary 2.2. *For each line l there are exact two half-planes bounded by l.*

(iii) *If A, B are on opposite sides of l and B, C are on the same side of l, then A, C are on opposite sides of l.*

Proof. Let A, B be two points on opposite sides of a line l. We have two distinct half-planes $H(l, A)$ and $H(l, B)$. Given an arbitrary point C not on l. If A, C are on the same side of l, then $H(l, C) = H(l, A)$. If A, C are on opposite sides, then B, C are on the same side of l by Betweenness Axiom 4(ii). Thus $H(l, C) = H(l, B)$. Therefore there are at most two half-planes bounded by l.

Given a point B on l and a point D not on l. By Betweenness Axiom 2 there exist points A, C, E such that $A * B * D$, $B * C * D$ and $B * D * E$. Then A, D are on opposite sides of l. So there are at least two half-planes bounded by l. □

Proposition 2.3 (Linearity rules). *Let A, B, C, D be distinct points on a line l. Then*

(a) $A * B * C, A * C * D \Rightarrow A * B * C * D$.

(b) $B * C * D, A * B * D \Rightarrow A * B * C * D$.

(c) $A * B * C, B * C * D \Rightarrow A * B * C * D$.

Proof. (a) Pick a point E outside l and make line $\overline{EC}$; see Figure 2. Then C is the unique

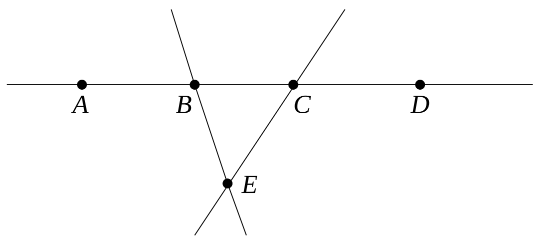

Figure 2: Betweenness and separation axioms imply linearity

intersection of l and $\overline{EC}$. The points A, B must be on the same side of line $\overline{EC}$. (Otherwise, AB meets $\overline{EC}$ at C; we then have $A*C*B$, which contradicts $A*B*C$.) Since $A*C*D$, then A, D are on opposite sides of $\overline{EC}$. Hence B, D are on opposite sides of $\overline{EC}$ by Corollary 2.2, i.e., BD meets $\overline{EC}$ at C. We then obtain $B * C * D$.

Draw line $\overline{EB}$; the point B is the unique intersection of l and $\overline{EB}$; see Figure 2. Since $A * B * C$, then A, C are on opposite sides of $\overline{EB}$. Since $B * C * D$, we must have C, D on the same side of $\overline{EB}$. (Otherwise B would be between C and D, contradicting to $B*C*D$.) Thus A, D are on opposite sides of $\overline{EB}$ by Corollary 2.2, i.e., AD meets $\overline{EB}$ at B between A and D. We then obtain $A * B * D$.

(b) is similar to (a) by reversing the order.

(c) Note that A, B, C are distinct and B, C, D are distinct. If $A = D$, then $B * C * D$ becomes $B * C * A$, which is contradictory to $A * B * C$. So A, B, C, D are distinct.

Pick a point E outside l and draw the line $\overline{EC}$. Since $B*C*D$, then B, D are on opposite sides of $\overline{EC}$ by definition. Likewise, $A * B * C$ implies that A, B are on the same side of $\overline{EC}$. (Otherwise, A, B are on opposite sides of $\overline{EC}$, i.e., AB meet $\overline{EC}$ at C; so $A * C * B$, contradicting to $A * B * C$.) It follows from Corollary 2.2 that A, D are on opposite sides of $\overline{EC}$. Hence AD meets $\overline{EC}$ at C between A and D, i.e., $A * C * D$. □

Definition 3 (Strict total order). A binary relation $\prec$ on a set X is called a **strict total order** if

(TO1) Irreflexivity: $x \not\prec x$ for all $x \in X$;

(TO2) Transitivity: if $x \prec y$ and $y \prec z$ then $x \prec z$;

(TO3) Completeness: either $x \prec y$ or $y \prec x$ but not both for all $x, y \in X$ with $x \neq y$.

For a strict total order on X, the relation $\preceq$, defined on X by $x \preceq y$ if $x = y$ or $\prec y$, is called a **total order**. For an order relation, we also write $x \prec y$ and $x \preceq y$ as $y \succ x$ and $y \succeq x$ respectively. The set X with a total order is said to be **totally ordered**.

Proposition 2.4 (Strict total order of line). *For each line l with two distinct points A, B there exists a unique total order on l such that $A \prec B$ and if $C * D * E$ then either*

$$C \prec D \prec E \quad \text{or} \quad E \prec D \prec C$$

but not both.

Proof. Define $A \prec B$. For each point P of l other than A, B, we define
(1) $P \prec A$ and $P \prec B$ if $P * A * B$,
(2) $A \prec P$ and $P \prec B$ if $A * P * B$,
(3) $A \prec P$ and $B \prec P$ if $A * B * P$.
For any two distinct points P, Q other than A and other than B, we define

$$P \prec Q \quad \text{if one of the following holds:}$$

(I) $P*Q*A*B$, (II) $P*A*Q*B$, (III) $P*A*B*Q$, (IV) $A*P*Q*B$, (V) $A*P*B*Q$, (VI) $A*B*P*Q$. We claim that $\prec$ is a strict total order on l.

It is clear that $\prec$ satisfies irreflexive and completeness. For transitivity, let $P \prec Q$ and $Q \prec R$, we claim $P \prec R$. If $\{P, Q, R\} \cap \{A, B\} \neq \emptyset$, we clearly have $P \prec R$ by definition of $\prec$. If $\{P, Q, R\} \cap \{A, B\} = \emptyset$, we verify the six cases.

Case I. $P * Q * A * B$.

(I.1) $Q*R*A*B$: Since $P*Q*A$ and $Q*R*A$, then $P*Q*R*A$ by Proposition 2.3(b). Since $P*R*A$ and $R*A*B$, then $P*R*A*B$ by Proposition 2.3(c). Hence $P \prec R$ by definition.

(I.2) $Q*A*R*B$: Since $P*Q*A$ and $Q*A*R$, then $P*Q*A*R$ by Proposition 2.3(c). Since $P*A*R$ and $A*R*B$, then $P*A*R*B$ by Proposition 2.3(c). Hence $P \prec R$ by definition.

(I.3) $Q*A*B*R$: Since $P*A*B$ and $A*B*R$, then $P*A*B*R$ by Proposition 2.3(c). By definition $P \prec R$.

Case II. $P * A * Q * B$.

(II.1) $A*Q*R*B$: Since $P*A*B$ and $A*R*B$, then $P*A*R*B$ by Proposition 2.3(b). By definition $P \prec R$.

(II.2) $A*Q*B*R$: Since $P*A*B$ and $A*B*R$, then $P*A*B*R$ by Proposition 2.3(c). By definition $P \prec R$.

Case III. $P * A * B * Q$.

(III.1) $A*B*Q*R$: Since $P*A*B$ and $A*B*R$, then $P*A*B*R$ by Proposition 2.3(c). By definition $P \prec R$.

Case IV. $A * P * Q * B$.

(IV.1) $A*Q*R*B$: Since $A*P*Q$ and $A*Q*R$, then $A*P*Q*R$ by Proposition 2.3(a). Since $P*Q*B$ and $Q*R*B$, then $P*Q*R*B$ by Proposition 2.3(b). We then have $A*P*R$ and $P*R*B$. Thus $A*P*R*B$ by Proposition 2.3(c). By definition $P \prec R$.

(IV.2) $A*Q*B*R$: Since $A*P*B$ and $A*B*R$, then $A*P*B*R$ by Proposition 2.3(a). By definition $P \prec R$.

Case V. $A * P * B * Q$.

(V.1) $A*B*Q*R$: Since $A*P*B$ and $A*B*R$, then $A*P*B*R$ by Proposition 2.3(a). By definition $P \prec R$.

Case VI. $A * B * P * Q$.

(VI.1) $A*B*Q*R$: Since $B*P*Q$ and $B*Q*R$, then $B*P*Q*R$ by Proposition 2.3(a). Since $B*P*R$ and $A*B*R$, then $A*B*P*R$ by Proposition 2.3. By definition $P \prec R$. $\square$

Proposition 2.5 (Line separation). *Let A, B, O be three distinct points such that $A{*}O{*}B$. Then*

$$r(O,A) \cap r(O,B) = \{O\}, \quad r(O,A) \cup r(O,B) = \overline{AB}.$$

If $P \in \overline{AB}$, then either $P \in r(O,A)$ or $P \in r(O,B)$. The rays $r(O,A)$ and $r(O,B)$ are said to be **opposite** *each other.*

Proof. Let $\prec$ be the strict total order on the line l such that $A \prec B$. By definition of the total order $\preceq$, the rays $r(O,A)$, $r(O,B)$, and the segment AB, we have

$$r(O,A) = \{P \in l : P \preceq O\}, \quad r(O,B) = \{P \in l : O \preceq P\}, \quad AB = \{P \in l : A \preceq P \preceq B\}.$$

Then $r(O,A) \cap r(O,B) = \{O\}$ and $r(O,A) \cup r(O,B) = \overline{AB}$ by the total ordering property of $\prec$. $\square$

Corollary 2.6 (Line separation). *Let l, m be two distinct lines intersecting at a point O. Let $\prec$ be a strict total order on l. Then the two sets*

$$\mathring{r}(O,-) := \{P \in l : P \prec O\}, \quad \mathring{r}(O,+) := \{P \in l : O \prec P\}$$

are on opposite sides of m. We also call them on **opposite sides** *of O on l.*

Proof. Let A, B be two distinct points on l. If $A, B \in \mathring{r}(O,-)$, i.e., $A \prec O, B \prec O$, then for all P between A and B, we have either $A \prec P \prec B$ or $B \prec P \prec A$. In either case we have $P \prec O$ by transitivity. So AB is contained in $\mathring{r}(O,-)$. Clearly, AB does not meet m (since O is the unique intersection of l and m). Hence A, B are on the same side of m by definition. Likewise, if $A, B \in \mathring{r}(O,+)$, i.e., $O \prec A, O \prec B$, then A, B are on the same side of m. If $A \prec O \prec B$ or $B \prec O \prec A$, then in either case AB meets m at O between A and B; so A, B are on opposite sides of m by definition. $\square$

Theorem 2.7 (Pasch's Theorem). *Let A, B, C be distinct points of not collinear. Let l be a line meeting AB at a point D between A and B. Then one and only one of the three holds: (i) l meets AC at a point between A and C, (ii) l meets BC at a point between B and C, (iii) l meets both AC and BC at a point C.*

Intuitively, this theorem says that if a line "goes into" a triangle through one side then it must "come out" through another side.

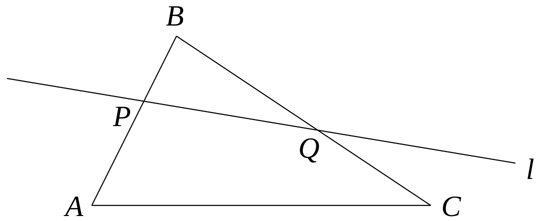

Figure 3: A line passes through a triangle

Proof. The points A, B are on the opposite sides of the line l. If C is on l, then l does not meet AC between A and C, otherwise $l = \overline{AC}$; and l does not meet BC between B and C. If C is not on l, then either A, C are on the same side of l, or B, C are on the same side of l, butt not both. In the formal case, then B, C are the opposite sides of l. Thus l meets BC at a point between meets B and C, and is disjoint from AC. In the latter case, l meets AC at a point between A and C, and is disjoint from BC. $\square$

Definition 4 (Interior of angle). Given points A, O, B not collinear. The **interior** of an angle $\angle AOB$, denoted $\mathring{\angle} AOB$, is the set of points P such that P, A are on the same side of line $\overline{OB}$, and P, B are on the same side of line $\overline{OA}$, in other words,

$$\mathring{\angle} AOB := \mathring{H}(\overline{OB}, A) \cap \mathring{H}(\overline{OA}, B);$$

see Figure 4. We also define

$$\angle AOB := H(\overline{OB}, A) \cap H(\overline{OA}, B).$$

It is convenient to consider a closed half-plane as a **flat angle**.

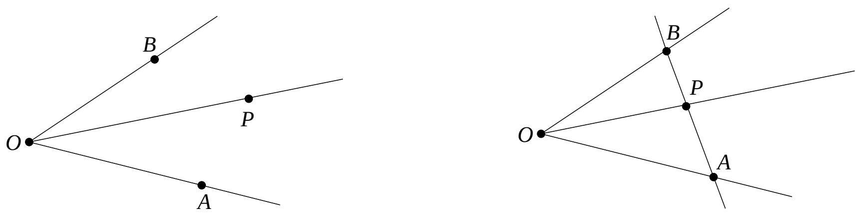

Figure 4: Interior of an angle

Proposition 2.8 (Between-Cross Lemma). *Given an angle $\angle AOB$ and a point P on $\overline{AB}$. Then P belongs to $\mathring{\angle} AOB$ if and only if $A * P * B$.*

Proof. "$\Rightarrow$": The point P belongs to $\mathring{\angle} AOB$. By definition P, B are on the same side of line $\overline{OA}$. Suppose $P * A * B$. Then P, B are opposite sides of $\overline{OA}$, since PB meets $\overline{OA}$ at A between P and B. This is a contradiction. Likewise, $A * B * P$ leads to a similar contradiction. Then we must have $A * P * B$ by trichotomy of betweenness.

"$\Leftarrow$": We have $A * P * B$. Note that line $\overline{AB}$ meets line $\overline{OB}$ at the unique point B. Then AP does not meet $\overline{OB}$. So A, P are on the same side of $\overline{OB}$. Likewise, points B, P are on the same side of $\overline{OB}$. Hence by definition P belongs to $\mathring{\angle} AOB$. $\square$

Proposition 2.9. *Let P be a point in $\mathring{\angle} AOB$. Then*

(a) The open ray $\mathring{r}(O, P)$ is contained in $\mathring{\angle} AOB$.

(b) The opposite ray to $r(O, P)$ is disjoint from $\mathring{\angle} AOB$. See the left of Figure 5.

(c) If $B * O * B'$, then A belongs to $\mathring{\angle} POB'$.

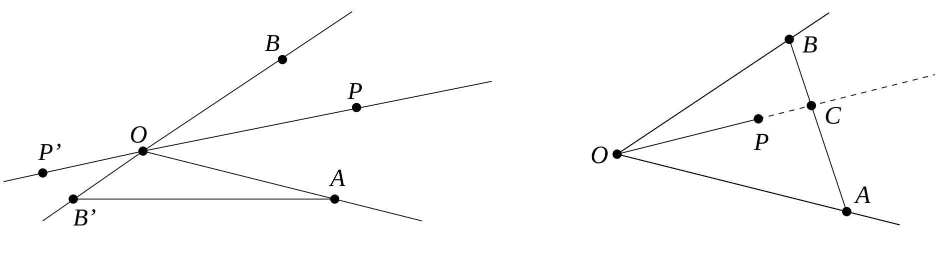

Figure 5: Property of interior of an angle and Crossbar Theorem

Proof. (a) Let Q be a point on the open ray $\mathring{r}(O, P)$. It is clear that PQ is disjoint from $\overline{OA}$ (since the intersection of the two lines $\overline{PQ}, \overline{OA}$ are the unique point O). This means that P, Q are on the same side of $\overline{OA}$ by definition. Since $P \in \mathring{\angle} AOB$, i.e., P, B are on the same side of $\overline{OA}$, then B, Q are on the same side of $\overline{OA}$. Likewise, A, Q are on the same side of $\overline{OB}$. Thus Q is an interior point of $\angle AOB$.

(b) Let P' be a point on the opposite ray of $r(O,P)$; see the left of Figure 5. Then P, P' are on opposite sides of $\overline{OB}$. Since A, P are the same side of $\overline{OB}$, then A, P' are on the opposite sides of $\overline{OB}$. Thus P' is not an interior point of $\angle AOB$ by definition.

(c) Note that P, A are on the same side of $\overline{OB'}$ (since $\overline{OB'} = \overline{OB}$ and $P \in \mathring{\angle} AOB$). We claim that A, B' are on the same side of $\overline{OP}$. If so, we have $A \in \mathring{\angle} POB'$ by definition.

Suppose that A, B' are on opposite sides of $\overline{OP}$, i.e., $\overline{OP}$ intersects AB' at C between A and B'. Then $A * C * B'$ and $C \in \mathring{\angle} AOB'$ by Proposition 2.8. Since $C \in \overline{OP}$ and $C \neq O$, we have either $C \in \mathring{r}(O,P)$ or $C \in \mathring{r}(O,P')$.

If $C \in \mathring{r}(O,P)$, then $P \in \mathring{r}(O,C)$, which is contained in $\mathring{\angle} AOB'$ by part (a). Thus P, B' are on the same side of $\overline{OA}$ (since $P \in \mathring{\angle} AOB'$). Since P, B are the same side of $\overline{OA}$, we see that B, B' are on the same side of $\overline{OA}$. This is a contradiction.

If $C \in \mathring{r}(O,P')$, then $P' \in \mathring{r}(O,C)$, which is contained in $\mathring{\angle} AOB'$ by part (a). Thus A, P' are on the same side of $\overline{OB}$ $(= \overline{OB'})$ by definition. Since P', P are on opposite sides of $\overline{OB}$, we see that A, P are on the opposite sides of $\overline{OB}$. This is a contradiction. □

Definition 5 (**Between rays).** A ray $r(O,P)$ is **between** two non-opposite rays $r(O,A)$ and $r(O,B)$ if P is in the interior of $\angle AOB$ (independent of the choice of P on the ray $r(O,P)$).

Proposition 2.10 (**Crossbar Theorem).** *If a ray $r(O,P)$ is between two rays $r(O,A)$ and $r(O,B)$, then $r(O,P)$ intersects AB at C between A and B. See the right of Figure 5. The interior of $\angle AOB$ is a disjoint union of interiors $\mathring{\angle} AOP$, $\mathring{\angle} BOP$, and open ray $\mathring{r}(O,P)$.*

Proof. Note that B, B' are on opposite sides of $\overline{OP}$, and B', A are on the same side of $\overline{OP}$; see the left of Figure 5. Then A, B are on opposite sides of $\overline{OP}$. Thus $\overline{OP}$ intersects AB. Since the ray $r(O,P')$ (opposite to the ray $r(O,P)$) is disjoint from the interior of $\angle AOB$, and since the open segment (A,B) is contained in the interior $\mathring{\angle} AOB$, then the open ray $\mathring{r}(O,P)$ must intersect AB at C between A and B; see the right of Figure 5. □

Definition 6 (**Interior of triangle).** The **interior** of a triangle ΔABC is the intersection of interiors of its three angles, denoted $\mathring{\Delta} ABC$. The **boundary** of ΔABC is the union of the three sides, i.e.,

$$\partial \Delta ABC := AB \cup AC \cup BC.$$

We also use ΔABC to denote the union of the interior and the boundary of ΔABC.

Proposition 2.11. *Given a triangle ΔABC and $O \in \mathring{\Delta} ABC$. Let $l = \overline{AB}$, $m = \overline{AC}$, $n = \overline{BC}$. Then*

(a) $\mathring{\Delta} ABC = \mathring{H}(l,O) \cap \mathring{H}(m,O) \cap \mathring{H}(n,O)$.

(b) Any ray $r(O,P)$ meets the boundary of ΔABC at a unique point Q.

Proof. (a) Trivial by $\mathring{\angle} ABC = \mathring{H}(l,O) \cap \mathring{H}(n,O)$ and other two interiors of angles.

(b) Let $l = \overline{OP}$. The line $\overline{OA}$ meet BC at D between B and C. We then have $A * O * D$, the open ray $\mathring{r}(D,O)$ is contained in $\mathring{H}(n,O)$, and its opposite half-line is contained in the opposite side of n. So $(A,D) := AD \setminus \{A,D\}$ is contained in the interior of ΔABC.

Case 1. $\overline{OP} = \overline{OA}$. Then $Q = A$ if $A * P * O$; $Q = D$ if $A * O * P$. See the left of Figure 6.

Case 2. $\overline{OP} \neq \overline{OA}$. The line $\overline{OA}$ separates the triangle ΔABC into two triangles ΔABD and ΔACD. Since $\overline{OP}$ meets AD at O between A and D, then $\overline{OP}$ meets the boundary of ΔABD at a unique point E and the boundary of ΔACD at a unique point F. Moreover, $E \in AB \cup BD$ and $F \in AC \cup CD$. If $r(O,P) = r(O,E)$, then $Q = E$. If $r(O,P) = r(O,F)$, then $Q = F$. □

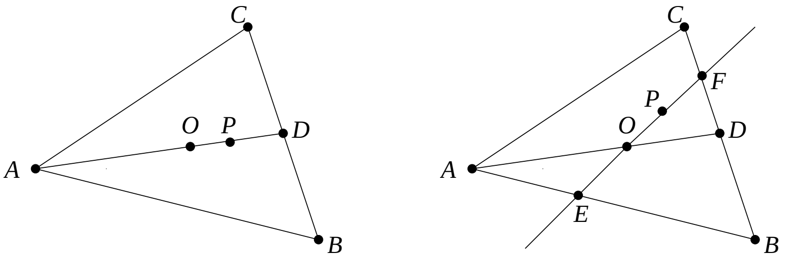

Figure 6: Ray emanating from the interior of a triangle

3 Axioms of Segment Congruence

Segments are not unrelated. We assume that there is a binary relation between segments, named as "**segment** AB **is congruent to segment** CD," abbreviated as

$$AB \cong CD.$$

Congruence Axiom 1 (CA1). Given two distinct points A, B, and a ray r emanating from a point A'. There exists exactly one point B' on r such that $B' \neq A'$ and $AB \cong A'B'$. Moreover, if $r = r(A, B)$, then $B' = B$; if $r = r(B, A)$, then $B' = A$.

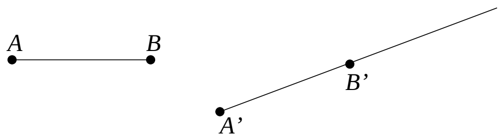

Figure 7: Congruence of segments

Congruence Axiom 2. (CA2) If $AB \cong CD$ and $CD \cong EF$, then $AB \cong EF$.

Proposition 3.1. *(1) $AB \cong AB$, $AB \cong BA$. (2) If $AB \cong CD$, then $CD \cong AB$.*

Proof. (1) It follows from the latter part of CA1. (2) Let $CD \cong AB'$, where B' is a point on the ray $r(A, B)$. Then $AB \cong AB'$ by transitivity. Hence $B' = B$ by CA1. We then have $CD \cong AB$. □

Congruence Axiom 3 (Segment addition) (CA3). If $A * B * C$, $A' * B' * C'$ and $AB \cong A'B', BC \cong B'C'$, then $AC \cong A'C'$.

Proposition 3.2 (Segment subtraction). *Given $A*B*C$ and $A'*B'*C'$. If $AB \cong A'B'$ and $AC \cong A'C'$, then $BC \cong B'C'$.*

Proof. Let $BC \cong B'P$, where P is a point on the ray $r(A', B')$. Then $AC \cong A'P$ by CA2. Since $AC \cong A'C'$, then $A'P \cong A'C'$ by CA2. Thus $P = C'$ by CA1. So $BC \cong B'C'$. □

Proposition 3.3 (Betweenness preserving by congruence of segments). *Given $AC \cong A'C'$ and $A * B * C$. Then there exists a unique point B' between A' and C' such that $AB \cong A'B'$ and $BC \cong B'C'$.*

Proof. Let $AB \cong A'B'$, where B' is the unique point on the ray $r(A', C')$. Let $BC \cong B'P$, where P is the unique point such that $A' * B' * P$. Since $AB \cong A'B'$ and $BC \cong B'P$, then $AC \cong A'P$ by CA3. Since $AC \cong A'C'$, then $P = C'$ by CA2. So $A' * B' * C'$. □

Proposition 3.4 (Congruence of lines). *For any two lines l and l', there exists a one-to-one correspondence $f : l \to l'$ such that $AB \cong f(A)f(B)$ for distinct points $A, B \in l$ and if $A * B * C$ then*

$$f(A) * f(B) * f(C).$$

Proof. Pick two points $O \in l$ and $O' \in m$. We have open rays $\mathring{r}(O,-), \mathring{r}(O,+)$ of l and open rays $\mathring{r}(O',-), \mathring{r}(O',+)$ of l'. Define $f(O) = O'$. For each $P \in \mathring{r}(O,-)$, there exists a unique point $P' \in \mathring{r}(O',-)$ such that $OP \cong O'P'$; define $f(P) = P'$. For each $Q \in \mathring{r}(O,+)$, there exists a unique point $Q' \in \mathring{r}(O',+)$ such that $OQ \cong O'Q'$; define $f(Q) = Q'$. We then have a map $f : l \to l'$. Likewise we have a map $f' : l' \to l$ defined in similar fashion. Then $f' \circ f : l \to l$ and $f \circ f' : l' \to l'$ are identity maps. So f and f' are bijections.

Given distinct points $A, B \in l$. If $A * B * O$ or $B * A * O$, then either $A, B \in \mathring{r}(O,-)$ or $A, B, \in \mathring{r}(O,+)$; thus $AB \cong f(A)f(B)$ by segment subtraction. If $A * O * B$, then $AB \cong f(A)f(B)$ by segment addition.

If $A * B * C$ on l, then there exists a unique point B'' be between $f(A)$ and $f(C)$ such that $AB \cong f(A)B''$ and $B''f(C)$ by the congruence of preserving betweenness. Since $AB \cong f(A)f(B)$, we must have $f(B) = B''$. Hence $f(A) * f(B) * f(C)$. □

Definition 7 (Linear order of segments). For segments AB, CD, if there exists a point E between C and D such that $AB \cong CE$, we write $AB < CD$ or $CD > AB$.

Theorem 3.5 (Strict total order of segments). *For two segments AB and CD, one and only one of the three holds: $AB < CD$, $AB \cong CD$, $AB > CD$ (**trichotomy**). Moreover,*

(a) If $AB \cong CD$ and $CD < EF$, then $AB < EF$.

(b) If $AB < CD$ and $CD \cong EF$, then $AB < EF$.

(c) If $AB < CD$ and $CD < EF$, then $AB < EF$.

Proof. Given segments AB and CD. Let $AB \cong CE$, where E is the unique point on the ray $r(C, D)$. We have one and only one of the three: $C * E * D$, $E = D$, $C * D * E$. These are exactly the three cases: $AB < CD$, $AB \cong CD$, $AB > CD$.

(a) Let P be a point such that $E * P * F$ and $CD \cong EP$. Then $AB \cong EP$ by CA2. Thus $AB < EF$ by definition.

(b) Let P be a point such that $C * P * D$ and $AB \cong CP$ by definition. Then there exists a point Q such that $E * Q * F$ and $CP \cong EQ$ by Proposition 3.3 (congruence of preserving betweenness). Then $AB \cong EQ$ by CA2. Thus $AB < EF$ by definition.

(c) Let P be such that $AB \cong CP$ and $C * P * D$. Let R be such that $E * R * F$ and $CD \cong ER$. Then there exists a point Q such that $E*Q*R$ and $CP \cong EQ$. Thus $AB \cong EQ$, $E * Q * R * F$, and of course $E * Q * F$. Therefore $AB < EF$. □

4 Axioms of Angle and Triangle Congruence

Angles are not unrelated. We assume that there is a binary relation between angles, named as "**angle $\angle ABC$ is congruent to angle $\angle DEF$**," abbreviated as

$$\angle ABC \cong \angle DEF.$$

Congruence Axiom 4 (CA4). Given an angle $\angle AOB$ and a ray $r(O', A')$, where the rays $r(O, A), r(O, B)$ are not opposite. There exists a unique ray $r(O', B')$ on each side of the line $\overline{O'A'}$ such that $\angle A'O'B' \cong \angle AOB$. Moreover, if $r(O', A') = r(O, A)$ and the side of $\overline{OA}$ is the half-plane $\mathring{H}(\overline{OA}, B)$, then $r(O, B') = r(O, B)$. If $r(O', A') = r(O, B)$ and the side of $\overline{OB}$ is the half-plane $\mathring{H}(\overline{OB}, A)$, then $r(O, B') = r(O, A)$.

Congruence Axiom 5 (CA5). If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

It is easy to see that angle congruence is reflexive, symmetric, and transitive. So angle congruence is an equivalence relation on angles.

Definition 8 (Congruence of triangles). A **triangle** is a collection of three non-collinear points A, B, C together with three segments AB, AC, BC (called **sides**), and three angles $\angle ABC, \angle ACB, \angle BAC$, denoted ΔABC. The point set of ΔABC, denoted by the same notation, is

$$\Delta ABC := \angle ABC \cap \angle ACB \cap \angle BAC.$$

The **interior** of ΔABC is the point set

$$\mathring{\Delta} ABC := \mathring{\angle} ABC \cap \mathring{\angle} ACB \cap \mathring{\angle} BAC.$$

Two triangles are said to be **congruent** if there is a one-to-one correspondence between their vertices such that the corresponding sides are congruent and the corresponding angles are congruent. More specifically, if a triangle with vertices A, B, C is congruent to a triangle with vertices A', B', C' by the one-to-one correspondence A to A', B to B', and C to C', written

$$\Delta ABC \cong \Delta A'B'C' \quad \text{(the order of vertices is material)},$$

then $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$, and $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$.

Congruence Axiom 6 (Side-angle-side) (SAS). If two sides and the included angle of a triangle are congruent respectively to two sides and the included angle of another triangle, then we say that the two triangles are **congruent**. More precisely, given two triangles with vertices A, B, C and vertices A', B', C'. If $AB \cong A'B'$, $AC \cong A'C'$, and $\angle A \cong \angle A'$, then $\Delta ABC \cong \Delta A'B'C'$.

Corollary 4.1. *Given a triangle ΔABC and a segment $A'B' \cong AB$. Then there exists a unique point C' on each side of the line $\overline{A'B'}$ such that $\Delta ABC \cong \Delta A'B'C'$.*

Proof. Choose a side of line $\overline{A'B'}$. There exists one and only one ray $r(A', P)$ such that $\angle B'A'P \cong \angle BAC$ by CA4. Then there exists a unique point C' on $r(A', P)$ such that $A'C' \cong AC$ by CA1. Thus $\Delta B'A'C' \cong \Delta BAC$ by SAS. □

Proposition 4.2. *Given a triangle ΔABC. If $AB \cong AC$, then $\angle B \cong \angle C$.*

Proof. Consider the one-to-one correspondence $A \leftrightarrow A$, $B \to C$, $C \to B$. We have $AB \cong AC$, $\angle BAC \cong \angle CAB$, $AC \cong AB$. Then $\Delta ABC \cong \Delta ACB$ by SAS. Thus $\angle B \cong \angle C$ by definition of congruence of triangles. □

Definition 9 (Supplementary angle, opposite angle, right angle). Supplementary angles and opposite angles are defined as before. A **right angle** is an angle which is congruent to its supplement. A closed half-plane is not an angle by our definition of angles; it is convenient to call it a **flat angle**.

Proposition 4.3 (Supplementary, opposite, right angle congruence rules). *(a) Supplements of congruent angles are congruent.*

(b) Opposite angles are congruent each other.

(c) Any angle congruent to a right angle is a right angle.

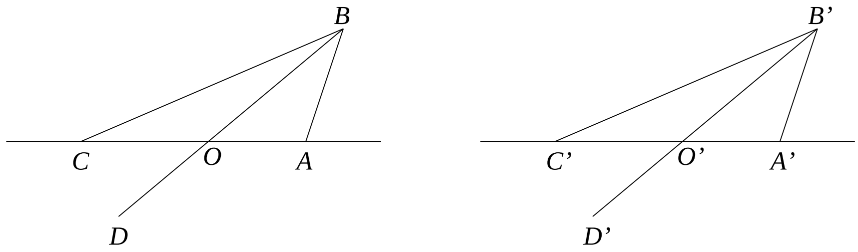

Figure 8: Supplements of congruence angles are congruent

Proof. Given two congruent angles $\angle AOB \cong \angle A'O'B'$. Pick a point C on the opposite ray of $r(O, A)$ with $C \neq O$. Pick a point C' on the opposite ray of $r(O', A')$ with $C' \neq O$. We may assume $OA \cong O'A'$, $OB \cong O'B'$, $OC \cong O'C'$. See Figure 8.

(a) We need to show $\angle BOC \cong \angle B'O'C'$. Since $OA \cong O'A'$, $\angle AOB \cong \angle A'O'B'$, $OB \cong O'B'$, then $\Delta AOB \cong A'O'B'$ by SAS. Then $AC \cong A'C'$ by CA3; $AB \cong A'B'$ and $\angle BAC \cong \angle B'A'C'$ by definition of congruence triangles. Thus $\Delta BAC \cong B'A'C'$ by SAS. Since $OC \cong O'C'$, $\angle OCB \cong O'C'B'$ and $CB \cong C'B'$, then $\Delta OCB \cong O'C'B'$ by SAS. We see $\angle BOC \cong \angle B'O'C'$.

(b) Consider opposite angles $\angle AOB$ and $\angle COD$ in the left of Figure 8. Both are supplementary to $\angle BOC$. So $\angle AOB \cong \angle COD$ by (a).

(c) Let $\angle AOB$ be a right angle. Need to show that $\angle A'O'B'$ is a right angle. Notice that $\angle AOB \cong \angle BOC$ by definition of right angles, $\angle B'O'C' \cong \angle BOC$ by (a), and $\angle AOB \cong \angle A'O'B'$ by given condition. Then $\angle A'O'B' \cong \angle B'O'C'$ by transitivity. This means that $\angle A'O'B'$ is a right angle. $\square$

Proposition 4.4 (Existence of perpendicular line). *For each line l and each point P not on l, there exists a unique line m through P perpendicular to l.*

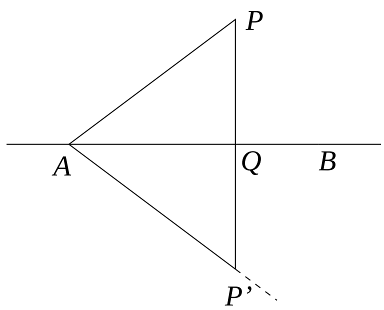

Figure 9: Construction of perpendicular lines

Proof. Pick two distinct points A, B on l. Draw segment AP. Then there exists a unique ray $r(A, C)$ on the opposite side of line l such that $\angle BAP \cong \angle BAC$. Mark a point P' on the ray $r(A, C)$ such that $AP \cong AP'$. Draw line $\overline{PP'} = m$. We claim that $m \perp l$. See Figure 9.

If A, P, P' are collinear, then A is the intersection of lines $\overline{AB}$ and $\overline{PP'}$. Clearly, $\angle BAP$ and BAP' are congruent supplementary angles. So they are right angles and $m \perp l$.

Assume that A, P, P' are not collinear. Since P, P' on opposite sides of l, then $r(P, P')$ intersects l at a unique point Q. We have triangles ΔAPQ and $\Delta AP'Q$. Since $AP \cong AP'$, $\angle PAQ \cong \angle P'AQ$, $AQ \cong AQ$, then $\Delta PAQ \cong \Delta P'AQ$ by SAS. Thus $\angle AQP \cong \angle AQP'$ by definition of congruence triangles, i.e., $m \perp l$. $\square$

Proposition 4.5 (**Angle-side-angle criterion) (ASA).** *If two angles and the included side of a triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.*

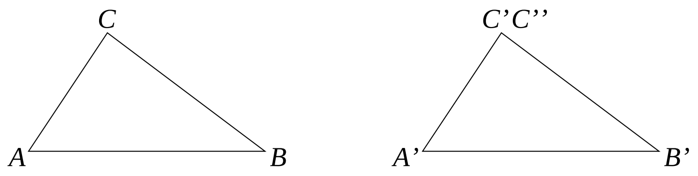

Figure 10: Angle-side-angle criterion

Proof. Given triangles ΔABC, $\Delta A'B'C'$, and $\angle BAC \cong \angle B'A'C'$, $AB \cong A'B'$, $\angle ABC \cong \angle A'B'C'$. Draw the unique point C'' on the ray $r(A', C')$ such that $AC \cong A'C''$. Then $\Delta ABC \cong \Delta A'B'C''$ by SAS. Then $\angle ABC \cong \angle A'B'C''$ by definition of congruence of triangles. Since $\angle ABC \cong \angle A'B'C'$ by given condition, then $\angle A'B'C' \cong \angle A'B'C''$ by transitivity. This means that B', C', C'' are collinear, i.e., C', C'' are on both lines $\overline{B'C'}$ and $\overline{A'C'}$. Since intersection point of two lines is unique, we have $C' = C''$. Hence $\triangle ABC \cong \triangle A'B'C'$. See Figure 10. Hence $\Delta ABC \cong \Delta A'B'C'$. □

Proposition 4.6 (**Angle addition).** *Given two angles $\angle AOC$ and $\angle A'O'C'$. Let $r(O, B)$ be a ray between rays $r(O, A)$ and $r(O, C)$. Let $r(O', B')$ be a ray between rays $r(O', A')$ and $r(O', C')$. If $\angle AOB \cong \angle A'O'B'$ and $\angle BOC \cong \angle B'O'C'$, then $\angle AOC \cong \angle A'O'C'$.*

Proof. We may assume that $OA \cong O'A'$, $OB \cong O'B'$, $OC \cong O'C'$, and that B is a point on AC between A, C. But we did not assume that B' is a point on $A'C'$. See Figure 11. Then $\Delta AOB \cong A'O'B'$ and $\Delta BOC \cong \Delta B'O'C'$ by SAS. We see that the supplementary angles $\angle OBA, \angle OBC$ are congruent to the angles $\angle O'B'A', \angle O'B'C'$ respectively. Then the supplementary angle $\angle O'B'C''$ of $\angle O'B'A'$ is congruent to $\angle OBC$ by the Supplementary Angle Congruence Rule. Thus $\angle O'B'C'' \cong \angle O'B'C'$ by transitivity. Since $\angle O'B'C''$ and $\angle O'B'C'$ are on the same side of line $\overline{O'B'}$, it follows that $\angle O'B'C'' = \angle O'B'C'$ by CA3. So A', B', C', C'' are collinear. Since $AB \cong A'B'$, $BC \cong B'C'$, then $AC \cong A'C'$. Since $\angle OAC \cong \angle O'A'C'$, $\angle OCA \cong \angle O'C'A'$, we have $\Delta AOC \cong \Delta A'O'C'$ by ASA. Therefore $\angle AOC \cong \angle A'O'C'$. □

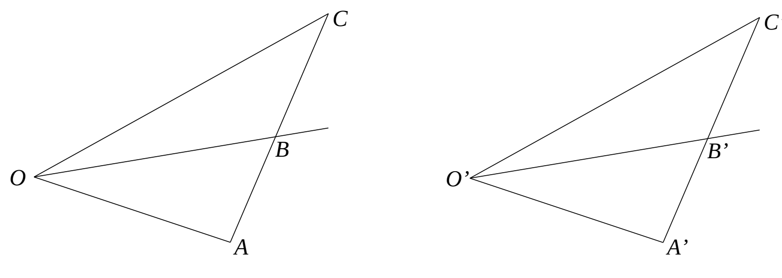

Figure 11: Angle subtraction

Proposition 4.7 (**Angle subtraction).** *Let $r(O, B)$ be a ray between rays $r(O, A)$ and $r(O, C)$. Let $r(O', B')$ be a ray between rays $r(O', A')$ and $r(O', C')$. If $\angle AOB \cong \angle A'O'B'$, $\angle AOC \cong \angle A'O'C'$, then $\angle BOC \cong \angle B'O'C'$.*

Proof. We may assume that $OA \cong O'A'$, $OC \cong O'C'$, and that B is a point on AC and B' is a point on $A'C'$. See Figure 11. Since $\angle AOC \cong \angle A'O'C'$, we have $\Delta AOC \cong \Delta A'O'C'$ by SAS. Thus $AC \cong A'C'$ and $\angle OAB \cong \angle O'A'B'$. Since $\angle AOB \cong \angle A'O'B'$, $OA \cong O'A'$, $\angle OAB \cong \angle O'A'B'$, then $\Delta OAB \cong \Delta O'A'B'$ by ASA. Thus $AB \cong A'B'$. Since $AC \cong A'C'$, then $BC \cong B'C'$ by Proposition 3.2 (segment subtraction). Now $OC \cong O'C'$, $\angle OCB \cong \angle O'C'B'$, $CB \cong C'B'$, we have $\Delta OCB \cong O'C'B'$ by SAS. Therefore $\angle BOC \cong \angle B'O'C'$. $\square$

Proposition 4.8. *Given a triangle ΔABC. If $\angle B \cong \angle C$, then $AB \cong AC$.*

Proof. Let $A \mapsto A$, $B \mapsto C$, $C \mapsto B$. Since $\angle ABC \cong \angle ACB$, $BC \cong CB$, $\angle ACB \cong \angle ABC$, then $\Delta ABC \cong ACB$ by ASA. Thus $AB \cong AC$ by definition of congruence of triangles. $\square$

Definition 10. An angle $\angle AOB$ is **less than** an angle $\angle A'O'C'$, written $\angle AOB < \angle A'O'C'$, if there exists a ray $r(O', B')$ between the rays $r(O', A')$ and $r(O', C')$, such that $\angle AOB \cong \angle A'O'B'$.

Proposition 4.9 (**Strict total order of angles**). *For any two angles $\angle A$ and $\angle B$, one and only one of the three holds: $\angle A < \angle B$, $\angle A \cong \angle B$, $\angle B < \angle A$ (**trichotomy**). Moreover,*

(a) If $\angle A \cong \angle B$, $\angle B < \angle C$, then $\angle A < \angle C$.
(b) If $\angle A < \angle B$, $\angle B \cong \angle C$, then $\angle A < \angle C$.
(c) If $\angle A < \angle B$, $\angle B < \angle C$, then $\angle A < \angle C$.

Proof. Given two angles $\angle AOB$ and $\angle A'O'B'$. There exists a unique open ray $\mathring{r}(O', C')$ in the open half-plane $\mathring{H}(\overline{O'A'}, B')$ such that $\angle AOB \cong \angle A'O'C'$. If C' is on the ray $r(O', B')$, then $r(O', C') = r(O', B')$ and $\angle AOB \cong \angle A'O'B'$. If C' is not on the ray $r(O', B')$, there are two cases.

Case 1. Points C', A' are on the same side of $\overline{O'B'}$.

Since C', B' are on the same side of $\overline{O'A'}$, then C' is contained in the interior of $\angle A'O'B'$; so is the open ray $\mathring{r}(O', C')$. Thus $\angle AOB < \angle A'O'B'$.

Case 2. Points C', A' are on opposite sides of $\overline{O'B'}$. Then $r(O', B')$ meets $A'C'$ at P' between A' and C' by Crossbar Theorem. Thus $\mathring{r}(O', B')$ is contained in $\mathring{\angle} A'O'C'$. By definition $\angle AOB > \angle A'O'B'$.

(a) Let $\angle AOB \cong \angle A'O'B' < \angle A''O''C''$. There exists a ray $r(O'', B'')$ between rays $r(O'', A'')$ and $r(O'', C'')$ such that $\angle A'O'B' \cong \angle A''O''B''$ by definition. Then $\angle AOB \cong \angle A''O''B''$ by transitivity. Thus $\angle AOB < \angle A''O''C''$.

(b) Let $\angle AOB < \angle A'O'C' \cong \angle A''O''C''$. There exists a ray $r(O', B')$ between the rays $r(O', A')$ and $r(O', C')$ such that $\angle AOB \cong \angle A'O'B'$. Let $r(O'', B'')$ be a ray between the rays $r(O'', A'')$ and $r(O'', C'')$ such that $\angle A'O'B' \cong \angle A''O''B''$. Then $\angle AOB \cong \angle A''O''B''$ by transitivity. Thus $\angle AOB < \angle A''O''C''$.

(c) Let $\angle AOB < \angle A'O'C' < \angle A''O''D''$. There exists a ray $r(O'', C'')$ between the rays $r(O'', A'')$ and $r(O'', D'')$ such that $\angle A'O'C' \cong \angle A''O''C''$. Then $\angle AOB < \angle A''O''C''$ by (b). Thus there exists a ray $r(O'', B'')$ between the rays $r(O'', A'')$ and $r(O'', C'')$ such that $\angle AOB \cong \angle A''O''B''$. Therefore $\angle AOB < \angle A''O''D''$. $\square$

Proposition 4.10 (**Side-side-side criterion**) **(SSS)**. *Given triangles ΔABC and $\Delta A'B'C'$. If $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$, then $\Delta ABC \cong \Delta A'B'C'$.*

Proof. Let C'' be the unique point on the opposite side of $\mathring{H}(\overline{A'B'}, C')$ bounded by $\overline{A'B'}$ such that $\Delta ABC \cong \Delta A'B'C''$. Draw the segment $C'C''$. The line $\overline{A'B'}$ meets $C'C''$ at D' between C' and C''. See Figure 12. Then $A'C' \cong AC \cong A'C''$ and $B'C' \cong BC \cong B'C''$, i.e., $\Delta A'C'C''$ and $\Delta B'C'C''$ are isosceles triangles. Hence $\angle A'C'C'' \cong \angle A'C''C'$ and $\angle B'C'C'' \cong \angle B'C''C'$.

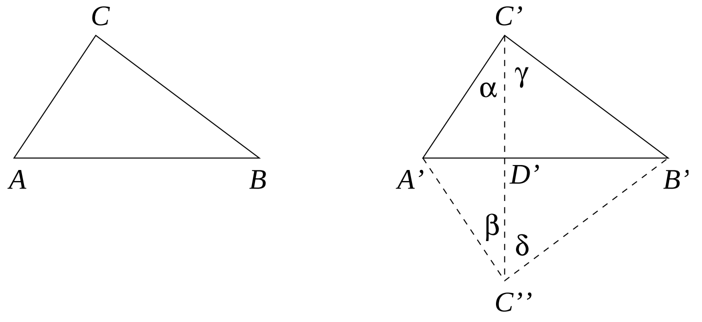

Figure 12: Side-side-side criterion

If $A' * D' * B'$, then the open ray $\mathring{r}(C', C'')$ is contained in $\mathring{\angle}A'C'B'$, and the open ray $\mathring{r}(C'', C')$ is contained in $\mathring{\angle}A'C''B'$ by Crossbar Theorem; thus $\angle A'C'B' \cong \angle A'C''B'$ by angle addition. If $A' * B' * D'$, then $\mathring{r}(C', B')$ is contained in $\mathring{\angle}A'C'D'$, and $\mathring{r}(C'', B')$ is contained in $\mathring{\angle}A'C''D'$ by Crossbar Theorem; thus $\angle A'C'B' \cong \angle A'C''B'$ by angle subtraction. Since $A'C' \cong AC \cong A'C''$, $\angle A'C'B' \cong \angle A'C''B'$, $B'C' \cong BC \cong B'C''$, we see that $\Delta A'B'C' \cong \Delta A'B'C''$ by SAS. Hence $\Delta ABC \cong \Delta A'B'C'$ by transitivity. $\square$

Theorem 4.11 (**Euclid’s Fourth Postulate**). *All right angles are congruent to each other.*

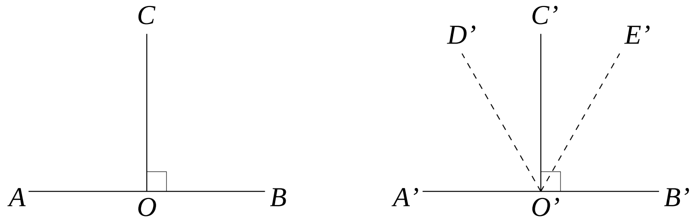

Figure 13: Euclid’s Fourth Postulate

Proof. Given angles $\angle AOC \cong BOC$ and $\angle A'O'C' \cong B'O'C'$; see Figure 13. We need to show $\angle AOC \cong \angle A'O'C'$. Let $\mathring{r}(O', P')$ be the unique open ray in $\mathring{H}(\overline{A'B'}, C')$ such that $\angle AOC \cong \angle A'O'P'$. It suffices to show that $\mathring{r}(O', P') = \mathring{r}(O', C')$. Suppose $\mathring{r}(O', P') \neq \mathring{r}(O', C')$. Then either $\mathring{r}(O', P') = \mathring{r}(O', D')$, which is contained in $\mathring{\angle}A'O'C'$, or $\mathring{r}(O', P') = \mathring{r}(O', E')$, which is contained in $\mathring{\angle}B'O'C'$.

In the former case, we have $\angle A'O'D' < \angle A'O'C'$ and $\angle B'O'C' < \angle B'O'D'$ by definition of order of angles. Since $\angle A'O'C' \cong \angle B'O'C'$ by right angle property, then $\angle A'O'D' < \angle B'O'D'$ by Proposition 4.9. Note that $\angle BOC \cong \angle B'O'D'$ by Proposition 4.3(a), and $\angle AOC \cong \angle A'O'D'$. Then $\angle AOC < \angle BOC$. However, $\angle AOC \cong \angle BOC$ by right angle property. In summary we have

$$\angle AOC \cong \angle A'O'D' < \angle A'O'C' \cong \angle B'O'C' < \angle B'O'D' \cong \angle BOC \cong \angle AOC.$$

So $\angle AOC < \angle AOC$; this is a contradiction. In the latter case, we have

$$\angle AOC \cong \angle A'O'E' > \angle A'O'C' \cong \angle B'O'C' > \angle B'O'E' \cong \angle BOC \cong \angle AOC.$$

So $\angle AOC > \angle AOC$; this is a contradiction. $\square$

5 Axioms of Continuity

Dedekind's Axiom (Continuity Axiom). If a line l is partitioned into two nonempty subsets Σ_1, Σ_2, i.e., $l = \Sigma_1 \cup \Sigma_1$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$, such that no point of either subset is between two points of the other (equivalently both are convex), then there exists a unique point O on l such that one of Σ_1, Σ_2 is a ray with vertex O and the other is an open ray with vertex O opposite to the other ray. The pair $\{\Sigma, \Sigma_2\}$ is called a **Dedekind cut** of l.

Given two distinct points $A, B \in l$. If $A, B \in \Sigma_i$, we have $AB \subset \Sigma_i$, i.e., Σ_i has no "hole." Suppose we do not require $\Sigma_1 \cap \Sigma_2 = \emptyset$ in Dedekind's axiom. If $A, B \in \Sigma_1 \cap \Sigma_2$, then we must have $A = B$. So the intersection $\Sigma_1 \cap \Sigma_2$ contains exactly one point O. Thus Σ_1, Σ_2 are two rays with the vertex O. So, when $\Sigma_1 \cap \Sigma_2 = \emptyset$ is imposed, we say that the partition $\{\Sigma_1, \Sigma_2\}$ determines one point on l.

Definition 11. A subset Ω of points is said to be **convex** provided that whenever two points P, Q are contained in Ω then the segment PQ is contained in Ω.

For a line l with a total order $\preceq$, the following subsets of l are convex sets, known as **intervals**: **line** l; **rays**

$$r(O, -) = \{P \in l : P \preceq O\}, \quad r(O, +) = \{P \in l : O \preceq P\};$$

open rays

$$\mathring{r}(O, -) = \{P \in l : P \prec O\}, \quad \mathring{r}(O, +) = \{P \in l : O \prec P\};$$

closed interval (segment)

$$[A, B] = AB = \{P \in l : A \preceq P \preceq B\};$$

open interval (segment)

$$(A, B) = \{P \in l : A \prec P \prec B\};$$

half-closed and half-open intervals (segments)

$$[A, B) = \{P \in l : A \preceq P \prec B\}, \quad (A, B] = \{P \in l : A \prec P \preceq B\}.$$

The points O, A, B are called **endpoints** of I. For a line not satisfying Dedekind's axiom, a convex subset of the line is not necessarily an interval.

Proposition 5.1 (**Extended Dedekind's Axiom).** *Dedekind's axiom is valid for any nonempty interval I of any line l. More precisely, if a nonempty interval I is partitioned into two nonempty convex sets Σ_1, Σ_2, then both Σ_1, Σ_2 are intervals with an endpoint O, one is closed and the other is open at O.*

Proof. Let l be totally ordered so that left and right sides of I are meaningful. If I is empty or contains exactly one point, nothing is to be proved for the statement is irrelevant. We assume that I contains at least two points.

Let Σ_1 be on the left side of Σ_2. Since I is nonempty, the complement $l \smallsetminus I$ has one of the forms: (a) empty set $\emptyset$, (b) a left (open) ray Γ_1, (c) a right (open) ray Γ_2, (d) disjoint union of a left (open) ray Γ_1 and a right (open) ray Γ_2.

Set $\Sigma_1' := \Sigma_1 \cup \Gamma_1$ and $\Sigma_2' := \Sigma_2 \cup \Gamma_2$. Then Σ_1', Σ_2' form a Dedekind cut of l. By Dedekind's axiom, there exists a unique point O on l such that Σ_1' is the right closed (open) ray $r(O, -)$ ($\mathring{r}(O, -)$) with vertex O, and Σ_2' is the left open (closed) ray $\mathring{r}(O, +)$ ($r(O, +)$) with vertex O. Hence Σ_1 is a right-closed (right-open) interval with right endpoint $O \in \Sigma_1$, and Σ_2 is left-open (left-closed) interval with left endpoint O. $\square$

Example. Let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{Q}^2$ forms an affine plane, called **rational affine plane** under its points and lines defined by one linear equation. Dedekind's axiom is not satisfied by the rational plane. Consider the x-axis $l = \{(a, 0) : a \in \mathbb{Q}\}$. Let $\Sigma_1 = \{(a, 0) : a \in \mathbb{Q}, a^2 > 3, a > 0\}$ and $\Sigma_2 = l \setminus \Sigma_1$. Then Σ_1, Σ_2 form a Dedekind cut, that is, they satisfy the conditions of Dedekind's axiom. However, neither Σ_1 nor Σ_2 is an (open) ray of l. In fact, $\Sigma_1 = \{(a, 0) : a \in \mathbb{Q}, a > \sqrt{3}\}$ is not an interval in $\mathbb{Q}^2$ (since it has no left endpoint).

Dedekind's axiom implies all of the following axioms.

Euclid's Proposition. For each segment there exists an equilateral triangle having one of its sides to be the given segment.

Definition 12. A point P is said to be **inside** a circle of radius OR with center O if $OP < OR$.

Circular Continuity Principle. If each of two circles has one point inside but outside the other, then the two circles intersect at two points.

Elementary Continuity Principle. If one endpoint of a segment is inside a circle and the other endpoint is outside, then the segment intersects the circle.

Archimedes'['a:ki'mi:di:z] Axiom. Given a segment AB and a ray r with vertex O. For each point $P \neq O$ on r, there exist an integer n and a point Q on r, where $OQ \cong n \cdot AB$, such that either $Q = P$ or $O * P * Q$.

Aristotle's['æristotl] Axiom. Given an acute angle $\angle AOB$ and a segment CD. There exists a point Y on the ray $r(O, B)$ such that $XY > CD$, where X is the foot of Y on the ray $r(O, A)$.

Proposition 5.2 (Dedekind's implies Archimedes'). *Dedekind's axiom implies Archimedes' axiom.*

Proof. Given a segment AB and a ray r with vertex O. A point $P \in r$ is said to be **reachable** by AB if $P = O$ or there exist an positive integer n and a point Q, such that $OQ \cong n \cdot AB$ and $O * P * Q$. Let Σ_1 be the set of points on r reachable by AB, and points on the opposite ray of r; so $O \in \Sigma_1$. Let Σ_2 be the complement of Σ_1 in the line l that contains r; Σ_2 is also the complement of Σ_1 in r. We claim that $\Sigma_2 = \emptyset$. (If so, all points on the ray r are reachable by AB, which is Archimedes' axiom.)

Suppose $\Sigma_2 \neq \emptyset$. We claim that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of l. One the one hand, let $P, Q \in \Sigma_1$ be distinct points. If both P, Q are on r or on the opposite ray of r, it is clear that $PQ \subset \Sigma_1$. If P is on the opposite ray of r and $Q \in r$, then $PO \subset \Sigma_1$ and $OQ \subset \Sigma_1$; so $PQ = PO \cup OQ \subset \Sigma_1$. On the other hand, let $P, Q \in \Sigma_2$ be distinct points. Suppose $PQ \not\subset \Sigma_2$; there exists a point $R \in \Sigma_1$ such that $P * R * Q$; since R can be reached, so is P; thus $P \in \Sigma_1$, which is a contradiction. Therefore $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of l, and determines a unique point O' on l.

Case 1. $O' \in \Sigma_1$. Then Σ_1 is a ray with vertex O'. Since the opposite ray of r is contained in Σ_1, then $O' \in r$ and Σ_2 is an open ray on r with vertex O'. Let O' be reached by laying off n copies of AB starting from O. Then by laying off one more copy of AB on r starting from O', we get points of Σ_2 being reached by AB. This is impossible.

Case 2. $O' \in \Sigma_2$. Then Σ_2 is a ray on r with vertex $O' \neq O$, and Σ_1 is the opposite open ray with vertex O'. Laying off one copy of AB starting from O' on the open ray Σ_1, we obtain a point P' in Σ_1. Then any point Q' such that $P' * Q' * O'$ is reachable by AB. Thus by laying one more copy of AB starting from Q', the point O' is reachable. So $O' \in \Sigma_1$. This is a contradiction. □

Proposition 5.3 (Dedekind's implies Elementary Continuity). *Dedekind's axiom implies Elementary Continuity Principle.*

Proof. Let γ be a circle with center O and radius OR. Let AB be a segment with A inside and B outside γ, i.e., $OA < OR$ and $OB > OR$. Let Σ_1 denote the set of points on AB inside γ, and Σ_2 the subset of points on AB outside or on γ. Then Σ_1, Σ_2 form a Dedekind cut for the segment AB by trichotomy of segments. Dedekind's axiom implies that there exists a unique point P on AB such that Σ_1, Σ_2 are intervals with endpoint P, one contains P and the other does not contain P. We claim that P is on γ, i.e., $OP \cong OR$.

Case 1. $OP < OR$. Then $P \in \Sigma_1$. Take a point $Q \in \Sigma_2$ such that $|PQ| = (|OR| - |OP|)/2$. By triangle inequality we have

$$|OR| < |OQ| < |OP| + |PQ| = |OP|/2 + |OR|,$$

which is a contradiction.

Case 2. $OP > OR$. Then $P \in \Sigma_2$. Take a point $Q \in \Sigma_1$ such that $A * Q * P$ and $|PQ| \leq (|OP| - |OR|)/2$. Since $|OQ| < |OR|$ and $|QP| = |PQ|$, then by triangle inequality

$$|OP| \leq |OQ| + |QP| < |OR| + (|OP| - |OR|)/2 = |OR|/2 + |OP|,$$

which is a contradiction.

So we must have $OP \cong OR$. $\square$

Relationship between the Axioms of Continuity.

Dedekind's axiom $\Rightarrow$ Archimedes' axiom, Circlar Continuity Principle

Archimedes' axiom $\Rightarrow$ Aristotle's axiom

Circlar Continuity Principle $\Rightarrow$ Elemenatry Continuity Principle